

# Balloon Calculations

Timothy Paul Smith

July 9, 2012

The purpose of this short note is to gather together all the equations and graphs which can be used to calculate the trajectory of a balloon, such as GreenCube or Altair. Typically we have used the *Edge of Space* software and website to do that. The derivation and equations reported here were originally “reversed engineered” to match their results, but then derived from first principles.

*Edge of Space* requires as input wind data, ascent rates, fall rates and burst height. The wind data changes hourly, and so must be obtained from NOAA. The other numbers, however, can be calculated. In the past we have relied upon tables provided by the balloon manufacture. We will derive those equations here.

The response of a balloon changes depending upon the conditions of the atmosphere. The properties reported by the balloon manufacture are calculated based upon the *Standard Atmosphere*.

We start with explaining the *Standard Atmosphere* in section 1, then the balloon parameters in section 2, the ind data in section 3, and finally the flight trajectory in section 4.

## 1 Standard Atmosphere

The Standard Atmosphere refers to a series of models of the atmosphere. There is a *International Standard Atmosphere* and the *US Standard Atmosphere*. They differ mainly in the latitude which they try to describe. The international model is for a more northern latitude, since Europe tends to be north of the US.

The US model has two versions, 1962 and 1976. So what is reported here is the *US Standard Atmosphere - 1976*

These models really are just a table of temperature, pressure and temperature graduate as a function of altitude. Actually the graduate can be derived from the temperature and altitude alone.

## 1.1 US Standard Atmosphere - 1976

height (km)	temp (°K)	press (atm)	temp-grad (°K/km)
0.00	288.15	1.00	-6.50
11.00	216.65	0.22	0.00
20.00	216.65	0.05	1.00
32.00	228.65	0.01	2.80
47.00	270.65	0.00	0.00
51.00	270.65	0.00	-2.80
71.00	214.65	0.00	-2.00
84.85	186.95	0.00	0.00

Table 1: US Standard Atmosphere - 1976

## 1.2 Zones

Each band of the atmosphere is called a *zone*. For example 0 to 11 km is zone 0, 51 to 71 km is zone 5.

## 1.3 Geopotential Altitude

The geopotential altitude is defined as

$$h = \frac{r_e z}{r_e + z}$$

where  $r_e$  is the radius of the earth and  $z$  is the real altitude. The purpose of this is that  $g$  - gravity ( $9.8 \text{ m/s}^2$ ) becomes smaller at high elevations because really

$$g(z) = \frac{Gm_e}{(r_e + z)^2}$$

With the geopotential altitude we have equation

$$g(z)dz = g_o dh$$

where  $g_o$  is  $g$  on the surface of the earth, ie 9.8 m/s<sup>2</sup>.

Thus  $h$  is used in hydrostatic equations AND  $h$  is the *height* in the table.

## 1.4 Calculating Temperature at any Altitude

The first step is always calculate the *geopotential altitude* which we will call  $h$  for height.

$$h = \frac{r_e z}{r_e + z}$$

Next we determine the *zone* from the table. This gives us the *base temperature* and the *temperature graduate*. From these we

$$T = T_{base} + dT dh$$

For example, the temperature at 30 km is

```
alt = 30;                // altitude (km)
rEarth = 6369.0;         // radius of the Earth (km)
h=alt*rEarth/(alt+rEarth);
    h = 29.859 km
    zone = 2
Tbase = 216.65;          // temp at the bottom of zone - from table
dT = 1;                  // temperature graduate - from table
dh = h - 20.0;          // distance from bottom of zone. bottom = 20 km - from table
T = Tbase + dT * dh;
    T = 226.5           // in deg-K
```

## 1.5 Calculating Pressure at any Altitude

The calculation of pressure is based on hydrostatic equations and the perfect gas laws and beyond the scope of this note. The results of the *pressure ratio* ( $p_r$ ) are reported here.

If  $dT$  is zero:

$$p_r = p_{base} e^{-(GMR dh/T_{base})}$$

else

$$p_r = p_{base} \left( \frac{T_{base}}{T} \right)^{(GMR/dT)}$$

with  $GMR = g_o M_o / R^* = 34.163195$

if (tgrad == 0.0)

delta=ptab[i]\*exp(-GMR\*deltah/tbase); //pressure ratio

else

delta=ptab[i]\*pow((tbase/tlocal),(GMR/tgrad));

NOTE: This is a *pressure ratio*. The pressure ( $p$ ) is then

$$p(alt) = p_r p(surface)$$

## 1.6 Calculating Density at any Altitude

Starting with the ideal gas law

$$P V = n R T$$

and with density

$$\rho \sim \frac{n}{V} = \frac{P}{R T}$$

Then the ratio is density at altitude to density on the surface is

$$\rho_r = \frac{p_r}{(T_{alt}/T_{base0})}$$

## 1.7 Standard Atmosphere as a Function of Altitude

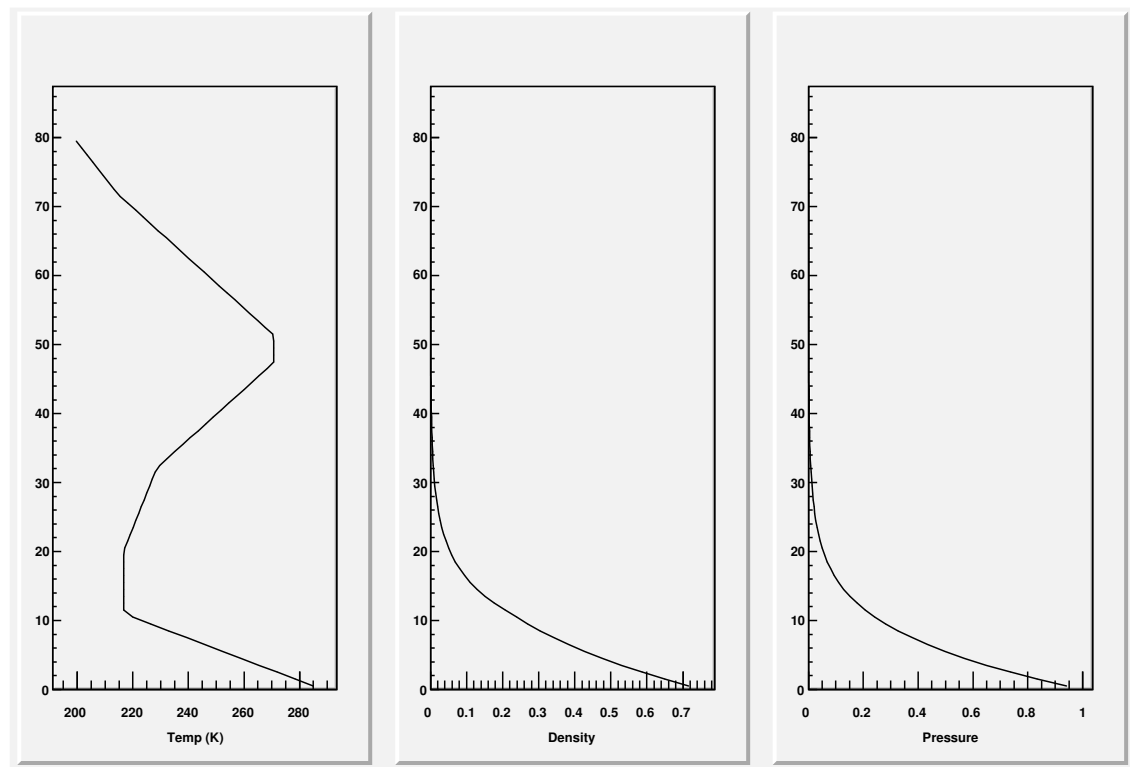


Figure 1: Standard Atmosphere as a Function of Altitude

## 2 Balloon Parameters

The origins of this problem is that the balloon manufacturer has supplied some table, but we would like to know the equations behind the numbers. First the tables



Figure 2: Balloon parameters for a (upper) 1200 gram balloon and (lower) 1500 gram balloon.

The input parameters are; temperature, balloon mass, payload mass, initial diameter and max diameter. From these we can calculate the; lift, free-lift, ascent-rate and the burst altitude.

The ascent-rate is velocity where the buoyancy (force) of the balloon

is balanced by the drag of the balloon raising through the air. Since this depends upon the air density, it will change with altitude. Here we calculate it at the surface of the Earth.

As the balloon raises, the pressure outside the balloon falls. Since the balloon is essentially in equilibrium, the pressure inside the balloon will also drop and the volume will increase. When the volume is greater than the maximum balloon diameter, it will bust. To do this calculation we need a model of the atmosphere, in order that we may know the density at different altitudes. That is part of the reason the *Standard Atmosphere* was introduced in section 1.

## 2.1 Ascent rate

The Ascent rate is velocity at which the buoyancy force is matched by the drag force.

### 2.1.1 Buoyancy

Start with the force due to gravity

$$F = mg$$

and the mass of the balloon, or the air which it displaces

$$m = \rho V$$

where  $V$  is the volume and  $\rho$  the density. Then the Buoyancy Force is

$$F_B = V(\rho_{He} - \rho_{air})g$$

Note: since the density of He is less than the density of Air, the force is negative, ie opposed to gravity (it raises).

### 2.1.2 Drag Force

Start with the drag equation

$$F_D = \frac{c_d \rho v^2 A}{2}$$

here  $c_d$  is the drag coefficient,  $\rho$  is the density of the medium (air/atmosphere),  $v$  is the velocity and  $A$  is the area of the balloon.

### 2.1.3 Solve for velocity

With  $V = (4/3)\pi r^3$  and  $A = \pi r^2$  we balance buoyancy and drag

$$\frac{4}{3}\pi r^3(\rho_{He} - \rho_{Air})g = \frac{c_d \rho v^2 \pi r^2}{2}$$

solve for  $v$

$$v = \sqrt{\frac{8r(\rho_{He} - \rho_{Air})g}{3c_d \rho_{Air}}} = \sqrt{\frac{4d(\rho_{He} - \rho_{Air})g}{3c_d \rho_{Air}}}$$

where  $d$  is the diameter of the balloon.

### 2.1.4 Coefficient of Drag

If you consult the literature, you will find that  $0.1 < c_d < 0.43$  for a sphere in air. That is a wide range and (because it appears inside of a square-root) will change the ascent rate by a factor of two. Ideally we could determine the coefficient based on the *Reynolds Number*. But the *Reynolds Number* is also hard to determine. It includes factors such as smoothness and friction.

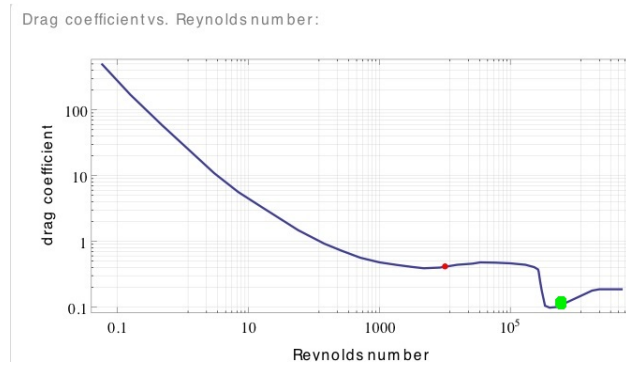


Figure 3: Drag coefficient of a sphere. Our case is near the green point.

An alternative method is to turn to balloons we have launched and the table which was supplied by the manufacturer. From these we can fit the coefficient to  $c_d = 0.285$ .



## 2.2 Burst Altitude

The Burst Altitude is the height where the size of the balloon exceeds its maximum size. The maximum size is determined by the manufacturer. The size of the balloon is how much helium is in it and the pressure. (pressure inside = pressure outside). The basic equation is the ideal gas law;

$$pV = nRT$$

$p$  = pressure,  $V$  = volume,  $n$  = amount,  $R$  = constant,  $T$  = temperature.

We re-write this as:

$$nR = \frac{pV}{T} = k$$

Since we know  $p$ ,  $V$  and  $T$  at the ground we also know  $nR = k$  at all altitudes. Then we can find  $V$  at any altitude, since we know  $p$  and  $T$ , based on our atmosphere model, at any altitude.

$$V_{alt} = \frac{kT_{alt}}{p_{alt}}$$

## 2.3 Burst Time

Burst time is essentially

$$t = \frac{\text{Burst Altitude}}{\text{Ascent Rate}}$$

## 2.4 New Tables

We can now produce the following tables:

1200 g balloon

diameter	volume	lift	maxAlt	radio	riseVel	burstTime	freeLift
[ft]	[ft <sup>3</sup> ]	[lbs]	[k ft]		[ft/min]	[hr]	[lbs]
7.1	187	9.49	97	rr	824	2.0	2.49
7.2	195	10.01	97	rr	893	1.8	3.01
7.3	204	10.55	96	rr	956	1.7	3.55
7.4	212	11.10	95	rr	1013	1.6	4.10
7.5	221	11.66	94	rr	1067	1.5	4.66

7.6	230	12.24	93	rr	1116	1.4	5.24
7.7	239	12.84	92	rr	1163	1.3	5.84
7.8	248	13.45	92	rr	1206	1.3	6.45
7.9	258	14.07	91	rr	1248	1.2	7.07
8.0	268	14.72	90	rr	1287	1.2	7.72
8.1	278	15.38	89	rr	1324	1.1	8.38
8.2	289	16.05	88	rr	1360	1.1	9.05
8.3	299	16.74	88	rr	1394	1.0	9.74
8.4	310	17.45	87	rr	1426	1.0	10.45
8.5	322	18.18	86	rr	1458	1.0	11.18

1500 g balloon

diameter	volume	lift	maxAlt	radio	riseVel	burstTime	freeLift
[ft]	[ft <sup>3</sup> ]	[lbs]	[k ft]		[ft/min]	[hr]	[lbs]
7.6	230	11.58	99	rr	1043	1.6	4.58
7.7	239	12.17	98	rr	1095	1.5	5.17
7.8	248	12.78	97	rr	1143	1.4	5.78
7.9	258	13.41	96	rr	1188	1.3	6.41
8.0	268	14.05	96	rr	1230	1.3	7.05
8.1	278	14.71	95	rr	1271	1.2	7.71
8.2	289	15.39	94	rr	1309	1.2	8.39
8.3	299	16.08	93	rr	1346	1.2	9.08
8.4	310	16.79	93	rr	1380	1.1	9.79
8.5	322	17.52	92	rr	1414	1.1	10.52
8.6	333	18.26	91	rr	1446	1.1	11.26
8.7	345	19.02	90	rr	1477	1.0	12.02

### 3 Wind Data

If you are going through the *Edge of Space* website it at one point jumps you into the [www.ready.noaa.gov](http://www.ready.noaa.gov) website, gets data and then has you cut and paste into back into the Edge of Space site.

You can go directly to the NOAA site via:

<http://www.ready.noaa.gov/READYcmet.php>

the Mount Washington Airport is: MWN

then select SOUNDING: GFS

which GFS depends upon how far in the future you are looking. Eventually you will find a table which is long (150-200 lines), but near the top looks like:

PRESS	HGT(MSL)	TEMP	DEW PT	WND DIR	WND SPD
HPA	M	C	C	DEG	M/S
E = Estimated Surface Height					
969.	378.	16.1	10.1	292.2	1.2
950.	548.	14.4	5.8	294.2	2.6
925.	773.	13.2	2.3	296.6	3.5
900.	1002.	11.2	1.2	300.9	4.7
850.	1476.	7.7	-2.7	305.4	8.6
800.	1973.	5.8	-11.0	306.3	11.3
750.	2499.	4.0	-17.4	301.4	12.0
700.	3057.	1.4	-18.3	294.1	12.9
650.	3649.	-2.0	-15.1	285.1	14.1
600.	4280.	-6.1	-17.5	277.3	15.3
550.	4956.	-9.6	-27.8	273.5	19.3
500.	5686.	-14.4	-33.8	271.8	25.4
450.	6475.	-20.1	-33.9	267.8	30.7
400.	7337.	-26.1	-41.2	266.6	34.4
350.	8291.	-32.1	-49.6	267.3	41.4
300.	9362.	-39.1	-53.8	265.3	49.2
250.	10596.	-44.4	-58.6	260.7	58.7
200.	12074.	-50.0	-61.6	264.1	63.9
150.	13934.	-53.2	-72.3	265.8	30.5
100.	16505.	-58.2	-83.3	269.8	14.6
50.	20921.	-52.7	-87.8	52.9	3.1
20.	26904.	-46.8	-273.1	75.6	12.8

Edge of space uses the whole table. My code uses only this part. The rest is the same information in a different format.

The density of air relative to that at the launch site is also important, but can be calculated from the above table data with

$$density = \frac{PRESS}{(TEMP + 273.2)}$$

## 4 Flight Simulation

Simulating the flight is now straightforward. The input parameters include primarily the wind table, the raise velocity, the fall velocity and the air density (calculated from the wind table).

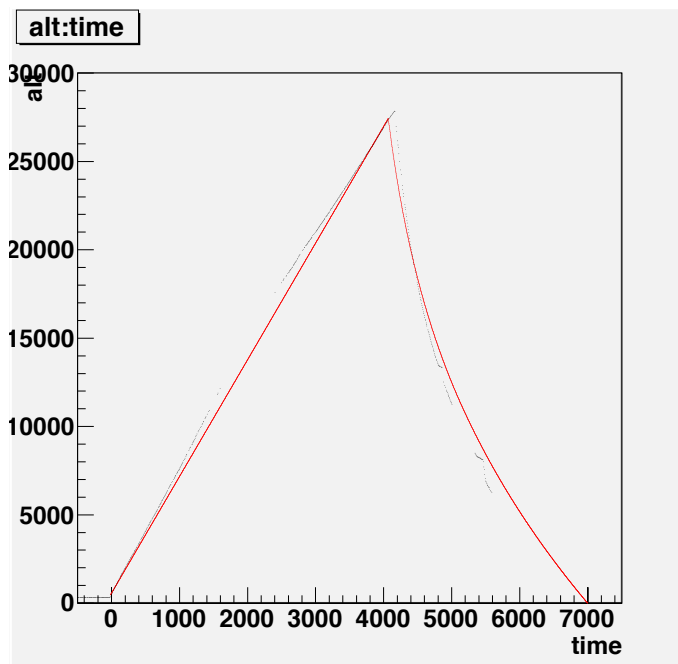


Figure 4: The Rise and Fall of a balloon. Black is real data, Red is calculated.

### 4.1 Rise

As can be seen from the above figure, the rise is nearly constant. We can treat it as constant, or re-calculate the rise velocity at every step in the ascent. The three-dimensional trajectory is simply integrating the path given the velocity up, and the wind velocity horizontally. What is more interesting is that during the rise the balloon passes through many bands of wind with

different directions. Thus the path is not simply downwind as the wind appears on the ground.

## 4.2 Fall

The fall is a bit more complex in that the fall rate changes with the density of the air. The number used in all the code is fall rate at 1 atmosphere. This can be scaled to the fall rate at any air density with:

$$fallRate = \frac{fallRate_o}{\sqrt{relDensity}}$$

where  $fallRate_o$  is the fall rate at 1 atmosphere and  $relDensity$  is the density relative to at the ground.